Tutorial 5 2022.10.26

5.1 Irrationality of $\zeta(3)$[1]

We give the proof for the irrationality of $\zeta(3)$. This proof is achieved by means of double and triple integrals, the shape of which is motivated by Apéry's formulas. Like Apéry's proof it also works for $\zeta(2)$, which is of course already known to be transcendental since it equals $\pi^2/6$. Most of the integrals that appear in the proof are improper. The manipulations with these integrals can be justified if one replaces $\int_0^1$ by $\int_\varepsilon^{1-\varepsilon}$ and by letting ε tend to zero.

Throughout this paper we denote the lowest common multiple of $1, 2, \ldots, n$ by d_n. The value of d_n can be estimated by

$$d_n = \prod_{\substack{\text{Prime} \\ p \leqslant n}} p^{[\log n/\log p]} < \prod_{\substack{\text{Prime} \\ p \leqslant n}} p^{\log n/\log p},$$

and the latter number is smaller than 3^n for sufficiently large n.

Lemma 5.1

Let r and s be non-negative integers. If $r > s$ then,

(a) $\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dxdy$ is a rational number whose denominator is a divisor of d_r^2.

(b) $\int_0^1 \int_0^1 -\frac{\log xy}{1-xy} x^r y^s dxdy$ is a rational number whose denominator is a divisor of ${d_r}^3$.

If $r = s$, then

(c) $\int_0^1 \int_0^1 \frac{x^r y^r}{1-xy} dxdy = \zeta(2) - \frac{1}{1^2} - \ldots - \frac{1}{r^2}$,

(d) $\int_0^1 \int_0^1 -\frac{\log xy}{1-xy} x^r y^r dxdy = 2\left\{\zeta(3) - \frac{1}{1^3} - \ldots - \frac{1}{r^3}\right\}$.

♡

Proof Let σ be any non-negative number. Consider the integral

$$\int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} dxdy \tag{5.1}$$

Develop $(1-xy)^{-1}$ into a geometrical series and perform the double integration. Then we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} \tag{5.2}$$

Assume that $r > s$. Then we can write this sum as

$$\sum_{k=0}^{\infty} \frac{1}{r-s} \left\{ \frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1} \right\} = \frac{1}{r-s} \left\{ \frac{1}{s+1+\sigma} + \ldots + \frac{1}{r+\sigma} \right\}. \tag{5.3}$$

If we put $\sigma = 0$ then assertion (a) follows immediately. If we differentiate with respect to σ and put $\sigma = 0$, then integral 5.1 changes into

$$\int_0^1 \int_0^1 \frac{\log xy}{1-xy} x^r y^s dxdy$$

and summation 5.3 becomes

$$\frac{-1}{r-s} \left\{ \frac{1}{(s+1)^2} + \ldots + \frac{1}{r^2} \right\}.$$

[1] This is a copy of Beukers, Frits. "A note on the irrationality of $\zeta(2)$ and $\zeta(3)$." Pi: A Source Book. Springer, New York, NY, 2004. 434-438.

Assertion (b) now follows straight away. Assume $r = s$, then by 5.1 and 5.2,

$$\int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy} dxdy = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)^2}.$$

By putting $\sigma = 0$ assertion (c) becomes obvious. Differentiate with respect to σ and put $\sigma = 0$. Then we obtain

$$\int_0^1 \int_0^1 \frac{\log xy}{1-xy} x^r y^r dxdy = \sum_{k=0}^{\infty} \frac{-2}{(k+r+1)^3},$$

which proves assertion (d).

Theorem 5.1

$\zeta(3)$ is irrational. ♡

Proof

Consider the integral

$$\int_0^1 \int_0^1 \frac{-\log xy}{1-xy} P_n(x) P_n(y) dxdy, \tag{5.4}$$

where $n! P_n(x) = \left\{\frac{d}{dx}\right\}^n x^n (1-x)^n$. It is clear from Lemma 5.1 that integral 5.4 equals $(A_n + B_n \zeta(3))\, d_n^{-3}$ for some $A_n \in \mathbb{Z}$, $B_n \in \mathbb{Z}$. By noticing that

$$\frac{-\log xy}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z} dz$$

integral (6) can be written as

$$\int \frac{P_n(x) P_n(y)}{1-(1-xy)z} dxdydz$$

where $\int$ denotes the triple integration. After an n-fold partial integration with respect to x our integral changes into

$$\int \frac{(xyz)^n (1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}} dxdydz \tag{5.5}$$

Substitute

$$w = \frac{1-z}{1-(1-xy)z}.$$

We obtain

$$\int (1-x)^n (1-w)^n \frac{P_n(y)}{1-(1-xy)w} dxdydw.$$

After an n-fold partial integration with respect to y we obtain

$$\int \frac{x^n (1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}} dxdydw.$$

It is straightforward to verify that the maximum of

$$x(1-x)y(1-y)w(1-w)(1-(1-xy)w)^{-1}$$

occurs for $x = y$ and then that

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leqslant (\sqrt{2}-1)^4 \text{ for all } 0 \leqslant x, y, w \leqslant 1.$$

Hence integral 5.4 is bounded above by

$$(\sqrt{2}-1)^{4n} \int \frac{1}{1-(1-xy)w} dxdydw = (\sqrt{2}-1)^{4n} \int_0^1 \int_0^1 \frac{-\log xy}{1-xy} dxdy = 2(\sqrt{2}-1)^{4n} \zeta(3).$$

Since integral 5.5 is not zero we have

$$0 < |A_n + B_n \zeta(3)|\, d_n^{-3} < 2\zeta(3)(\sqrt{2}-1)^{4n}$$

and hence

$$0 < |A_n + B_n\zeta(3)| < 2\zeta(3)d_n^3(\sqrt{2}-1)^{4n} < 2\zeta(3)27^n(\sqrt{2}-1)^{4n} < \left(\frac{4}{5}\right)^n$$

for sufficiently large n, which implies the irrationality of $\zeta(3)$.